PMM U.S.S.R.,Vol.46,pp.161-166 0021-8928/83/2 0161 $7.50/0

UDC 531.34

# METHOD OF MOMENTS IN PROBLEMS OF DYNAMICS OF SYSTEMS WITH RANDOMLY VARYING PARAMETERS*

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Derivation of general equations of the method of moments for linear systems with random variation of parameters is presented on the basis of Markovian theory of diffusion processes. As an example, a system with a single degree of freedom with random variation of its natural frequency with periodic or random external action, and a system with two degree of freedom under conditions of stochastic combination resonance are considered. These examples are used for demonstrating the method of moments in conjunction with that of averaging.

The method of moments, used for deriving from equations of motion a system of deterministic equations with respect to first and second moments of variables of state, are widely applied in the case of linear systems with random external actions and deterministic laws of parameters variation with time /1,2/, as well as for analyzing the stability of systems with random parameter excitation /1,3/. It seems that only some particular problems were considered /4/ in the case of systems with simultaneous external and parametric excitation.

1. Equations of the method of moments are derived on the initial assumption that in a linear system the random parametric excitations are normal random processes of the white noise type, uncorrelated with random external effects. The latter are also assumed to be normal and can be expressed in terms of white noise processes using auxilliary systems of differential equations that form filters. With the indicated constraints the linear system of general form can be defined by the following system of stochastic differential equations (taken here in the meaning of Stratonovich /3,5/):

$$x_i^{\cdot} = \sum_{j=1}^{n} b_{ij}x_j + \sum_{j=1}^{n}\sum_{r=1}^{s} \sigma_{ijr}x_j\xi_r(t) + f_i(t) + \sum_{r=1}^{n} \gamma_{ir}\zeta_r(t), \quad i = 1,\ldots,n; \quad s \leqslant n^2 \tag{1.1}$$

where $\xi_r(t)$, $\zeta_r(t)$ are independent stationary centered normal processes of the white noise type of unit intensities (in the case of input systems with correlated perturbations, linear combinations of the latter are introduced, as shown in /3/). The coefficients $b_{ij}$, $\sigma_{ijr}$, $\gamma_{ir}$ can, generally, be time dependent functions, and functions $f_i(t)$ are also determinate. Initial values of variables $x_i(t)$ of state are assumed to be either determinate or statistically independent with respect to $\xi_r(t)$, $\zeta_r(t)$.

The basis of the method of moments is the property of noncorrelation of the (vector) process that satisfies Ito's stochastic differential equation and the vector of random excitations appearing in that equation. It is, consequently, necessary to pass from Eqs.(1.1) in conformity with known relations to the following system of stochastic equations in Ito's meaning /3,5/:

$$x_i^{\cdot} = \sum_{j=1}^{n} b_{ij}x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{r=1}^{s} \sigma_{ilr}\sigma_{ljr}x_j + f_i(t) + \sum_{j=1}^{n}\sum_{r=1}^{s} \sigma_{ijr}x_j\xi_r(t) + \sum_{r=1}^{n} \gamma_{ir}\zeta_r(t) \tag{1.2}$$

Applying to (1.2) the operation of mathematical expectation determination (denoted below by angle brackets), we obtain for the $n$-dimensional vector $m(t)$ of the first moments $m_i(t) = \langle x_i(t)\rangle$ of the variables of state the following deterministic equation:

$$m = \left[B + \frac{1}{2}\sum_{r=1}^{s} \sigma_r^2\right] m + f(t) \tag{1.3}$$

$$B = \| b_{ij} \|, \quad \sigma_r = \| \sigma_{ijr} \|, \quad f(t) = \| f_i(t) \|$$

Equations for the second moments of variables of state $K_{ik}(t) = \langle x_i(t)\ x_k(t)\rangle$ may be derived in two ways. First, it is possible to introduce the variables $u_{ik} = x_ix_k$ and determine derivatives $u_{ik} = x_i^{\cdot}x_k + x_k^{\cdot}x_i$ by virtue of Eqs.(1.2) in conformity with Ito's differentiation formula

*Prikl.Matem.Mekhan.,46,No.2,pp.218-224,1982

/3,6/. It is also possible to determine $u_{ik}^{\cdot}$ using the Stratonovich equations (1.1) in conformity with the "usual" rules of substitution of variables, and then pass from the obtained equations in $u_{ik}$ to the respective Ito's equations. Application to the latter of the operation of mathematical expectation determination results in both cases in the following system of equations with respect to second order moments:

$$K_{ik}^{\cdot} = \sum_{j=1}^{n} (b_{ij}K_{jk} + b_{kj}K_{ij}) + \sum_{r=1}^{s} \sum_{l=1}^{n} \sum_{j=1}^{n} [\sigma_{kjr}\sigma_{ilr}K_{jl} + {}^1/_2 (\sigma_{ilr}\sigma_{ljr}K_{jk} + \sigma_{ljr}\sigma_{klr}K_{ij})] + \sum_{r=1}^{n} \gamma_{ir}\gamma_{kr} + f_i m_k + f_k m_i \tag{1.4}$$

where the first order moments $m_i$ and $m_k$ are determined by the integration of Eqs.(1.3).

If the input system of equations contains a small parameter, it is possible to considerably reduce the imposed constraints (particularly those related to the Gaussian properties of processes $\xi(t)$, $\zeta(t)$). For this the theorems on the asymptotically Markovian properties of solutions of such equations /5,7/ should be used together with the application of the described above algorithm for composing equations of moments to the respective limit vector diffusion process. With a finite small parameter such asymptotic approach, although approximate,has a number of advantages. First, it enables us to substantially increase the class of systems that can be analyzed by the method of moments. Its use does not necessarily require to consider parametric perturbations $\xi(t)$ as processes of the white noise type (although they must be of the wide-band type). This makes it possible to analyze the effects associated with various spectral densities of this or that perturbation $\xi(t)$ at various characteristic frequencies of the system. Second, the application of the Krylov—Bogoliubov method of averaging in conformity with the theorem in /7/ enables us to eliminate the periodic dependence of solution on time, thus substantially simplifying the equations of moments. Examples of application of this method are given below. They also show that its use enables us to obtain closed systems of equations for first and second order moments, as well as in the case of some specific nonlinear dependence for diffusion coefficients of the Markovian vector process (nonlinearities of the square root type).

2. Consider the system with one degree of freedom

$$x^{\cdot\cdot} + 2\alpha x^{\cdot} + \Omega^2 x [1 + \xi(t)] = y(t) \tag{2.1}$$

where $\xi(t)$ is a stationary wide-band random process, and parameters $\alpha$, $\Omega$ are constant. Let initially $y(t) = \zeta(t)$, where $\zeta(t)$ is a wide-band random process, and spectral densities $\Phi_{\xi\xi}(\omega)$, $\Phi_{\zeta\zeta}(\omega)$ of processes $\xi(t)$, $\zeta(t)$ and, also, parameter $\alpha$ are assumed small. It is then possible to introduce in (2.1) the substitution of variables

$$x = A(t) \cos \theta(t), \ x^{\cdot} = -\Omega A(t) \sin \theta(t), \ \theta = \Omega t + \varphi(t)$$

The application of the method of averaging in conformity with the theorem in /7/ yields for the amplitude $A(t)$ the following abbreviated stochastic equation in the sense of Ito /2/:

$$A^{\cdot} = -\alpha A + \frac{\pi\Phi_{\zeta\zeta}(\Omega)}{2\Omega^2 A} + \frac{3}{8}\pi\Omega^2\Phi_{\xi\xi}(2\Omega) A + \eta_\xi(t) + \eta_\zeta(t). \tag{2.2}$$

where $\eta_\zeta(t)$, $\eta_\xi(t)$ are equivalent processes of the white noise type with intensity coefficients $\pi\Phi_{\zeta\zeta}(\Omega)/\Omega^2$ and ${}^1/_4\pi\Omega^2\Phi_{\xi\xi}(2\Omega)A^2$, respectively. We substitute the variable $V = A^2$ in (2.2) and apply Ito's formula. As the result we obtain for the mathematical expectation $m_V = \langle V \rangle$ of the square of amplitude the equation

$$m_V^{\cdot} = -(2\alpha - \pi\Omega^2\Phi_{\xi\xi}(2\Omega))\, m_V + 2\pi\Phi_{\zeta\zeta}(\Omega)/\Omega^2 \tag{2.3}$$

which is readily integrable in explicit form. When $\alpha < {}^1/_2\pi\Omega^2 \Phi_{\xi\xi}(2\Omega)$, the solution increases indefinitely as $t \to \infty$ (system (2.1) is unstable in the mean square), and in the case of $\alpha > {}^1/_2\pi\Omega^2 \Phi_{\xi\xi}(2\Omega)$ with $t \to \infty$ we have the stationary solution

$$m_V = \frac{2\pi\Phi_{\zeta\zeta}(\Omega)}{\Omega^2(2\alpha - \pi\Omega^2\Phi_{\xi\xi}(2\Omega))}$$

A direct application of the method of moments to Eq.(2.1) (on the assumption that $\xi(t)$, $\zeta(t)$ are independent normal processes of the white noise type with respective intensity coefficients $D_\xi = 2\pi\Phi_{\xi\xi}(2\Omega)$, $D_\zeta = 2\pi\Phi_{\zeta\zeta}(\Omega)$) yields a system of three equations in second moments $\langle x^2 \rangle$, $\langle x^{\cdot 2} \rangle$, $\langle xx^{\cdot} \rangle$ /4/. When $\alpha$, $D_\xi$, $D_\zeta$ are small, the slowly varying principal part of solution of this exact system of equations is of the form $\langle x^2\rangle = \langle x^{\cdot 2}\rangle/\Omega^2 = {}^1/_2 m_V$, where $m_V$ is the solution of Eq.(2.3). It is interesting that the application of the theorem about the asymptotically Marcovian properties of the process $A(t)$ enable us to disregard the initial assumption of absence of correlation between $\xi(t)$ and $\zeta(t)$, since in conformity with (2.2) in the approximation of the asymptotic method the effect of that correlation is immaterial.

Let now in (2.1) $y = a\cos\nu t$, where $|\Delta|$ and $a$, $\Phi_{\xi\xi}(\omega)$, $\alpha$ are smalls of the same order ($\Delta = \nu - \Omega$). We pass in (2.1) to the new slow variables $x_c(t)$, $x_s(t)$ using formulas

$$x = x_c\cos\nu t + x_s\sin\nu t, \quad x^{\cdot} = \nu(-x_c\sin\nu t + x_s\cos\nu t) \tag{2.4}$$

Then from (2.1) we obtain the following system of equations in $x_c(t)$, $x_s(t)$:

$$x_c^{\cdot} = -\alpha x_c - \Delta x_s + {}^1/_2\,\Omega\,[x_c\sin 2\nu t + x_s(1 - \cos 2\nu t]\,\xi(t), \quad x_s^{\cdot} = -\alpha x_s + \Delta x_c + a/2\nu - {}^1/_2\,\Omega\,[x_c(1 + \cos 2\nu t) + x_s\sin 2\nu t]\,\xi(t) \tag{2.5}$$

in which averaging over the period of terms not containing $\xi(t)$ has been already carried out to abbreviate recording, and terms of higher order of smallness rejected.

Applying the theorem in /7/ to the system of Eqs.(2.5) we obtain two stochastic Ito's equations with respect to components of the limit two-dimensional Markovian process $x_c(t)$, $x_s(t)$. The subsequent application of the operation of mathematical expectation determination yields the following system of deterministic equations in $m_{c,s}(t) = \langle x_{c,s}(t)\rangle$:

$$\begin{aligned} m_c^{\cdot} &= -m_c[\alpha - {}^1/_8\,\Omega^2(D_2 - D_0)] + m_s(\Delta - {}^1/_{16}\,\Omega^2 E) \\ m_s^{\cdot} &= -m_c(\Delta - {}^1/_{16}\,\Omega^2 E) - m_s[\alpha - {}^1/_8\,\Omega^2(D_2 - D_0)] + a/2\nu \end{aligned} \tag{2.6}$$

where

$$D_2 = \int_{-\infty}^{\infty} K_{\xi\xi}(\tau)\cos 2\nu\tau\,d\tau = 2\pi\Phi_{\xi\xi}(2\nu) \approx 2\pi\Phi_{\xi\xi}(2\Omega) \tag{2.7}$$

$$D_0 = 2\pi\Phi_{\xi\xi}(0), \quad E = 2\int_0^{\infty} K_{\xi\xi}(\tau)\sin 2\nu\tau\,d\tau$$

and $K_{\xi\xi}(\tau)$ is the correlation function of process $\xi(t)$.

The stationary solution ($m_{c,s}^{\cdot} = 0$) of the system of Eqs.(2.6) with constant $a$ and $\nu$ is of the form

$$\begin{aligned} m_s &= (a/(2\nu P))[\alpha + {}^1/_8\,\Omega^2(D_0 - D_2)] \\ m_c &= (a/(2\nu P)(\Delta - {}^1/_{16}\,E) \\ P &= [\alpha + {}^1/_8\,\Omega^2(D_0 - D_2)]^2 + (\Delta - {}^1/_{16}\,E)^2 \end{aligned} \tag{2.8}$$

If the quantities $D_0, D_2, E$ are smalls of higher order than $\alpha$, it is possible to derive from (2.8) the expression for the quantity $\langle A\rangle = \max_t\langle x(t)\rangle = (m_c^2 + m_s^2)^{1/2}$, which was earlier obtained in /8/ by the method of perturbations (more exactly, we are concerned here with the limit variant of the expression obtained in /8/ for $\tau_0 \ll 1/\alpha$, where $\tau_0$ is the correlation time of process $\xi(t)$). That expression shows (see also /2/) that when $\Delta = 0$, the quantity $\langle A\rangle/A_0 - 1$ is proportional to the remainder of $[\Phi_{\xi\xi}(0) - \Phi_{\xi\xi}(2\Omega)]$, where $A_0 = a/(2\nu\alpha)$ is the amplitude of steady periodic oscillations of system (2.1) when $\xi(t) \equiv 0$. This was the basis of the conclusion in /8/ on the theoretical feasibility of lowering the amplitude of mechanical systems resonance oscillations by the introduction of special slow fluctuations of the natural frequency. However the method of perturbations has only a limited application range and, consequently, the detailed analysis of that method of reducing vibrations was carried out in /8/ by analog simulation.

The method of moments considerably increases the scope of theoretical analysis in comparison with that of perturbations. Using (2.5) we compose the equations of second moments

$$K_{cc} = \langle x_c^2\rangle, \quad K_{ss} = \langle x_s^2\rangle, \quad K_{cs} = \langle x_c x_s\rangle$$

of processes $x_c(t)$, $x_s(t)$. On the basis of (2.5) we write expressions for derivatives of $x_c^2$, $x_s^2$, $x_c x_s$, then, using the theorem of /7/, apply to the obtained system of three shortened Ito's equations the operation of mathematical expectation determination, and obtain

$$\begin{aligned} K_{cc}^{\cdot} &= -[2\alpha + {}^1/_8\,\Omega^2(2D_0 - 3D_2)]K_{cc} + {}^1/_8\,\Omega^2(2D_0 + D_2)\,K_{ss} - 2\Delta K_{cs} \\ K_{ss}^{\cdot} &= -[2\alpha + {}^1/_8\,\Omega^2(2D_0 - 3D_2)] + {}^1/_8\,\Omega^2(2D_0 + D_2)\,K_{cc} + 2\Delta K_{cs} + am_s/\nu \\ K_{cs}^{\cdot} &= -[2\alpha + {}^1/_4\Omega^2(2D_0 - D_2)]\,K_{cs} + (\Delta - {}^1/_8\Omega^2 E)(K_{cc} - K_{ss}) \end{aligned} \tag{2.9}$$

whose stationary solution with constants $a$ and $\nu$ yields

$$\langle x_s^2 \rangle + \langle x_c^2 \rangle = \langle A^2 \rangle = \frac{am_s/\nu}{2\alpha - {}^1/_2\Omega^2 D_2} \tag{2.10}$$

Formulas (2.10) and (2.8) show that, when $D_2, D_0, E$ are of a higher order of smallness than $\alpha$, then with $\Delta = 0$ the quantity $\langle A^2 \rangle / A_0^2 - 1$ is proportional to the remainder $[\Phi_{\xi\xi}(0) - 3\Phi_{\xi\xi}(2\Omega)]$. Moreover, it follows from (2.10) and (2.8) that when $\Phi_{\xi\xi}(\omega)$ is proportional to some positive parameter $\mu$ and $\Phi_{\xi\xi}(0) > 3\Phi_{\xi\xi}(2\Omega)$, then, as $\mu$ increases, the mean square of amplitude $\langle A^2 \rangle$ first decreases (if $\Delta = 0$)) and then begins to increase. This effect revealed in /8/ using analog simulation, and indicating limitations of that method of vibration reduction, cannot be defined in the perturbation method (it is associated with the approach to stability limit in the mean square defined by the equality $D_2 = 4\alpha / \Omega^2$).

An expression similar to (2.10) obtains for $\langle A^2 \rangle$, when in (2.1) $y(t)$ is a narrow-band random process that satisfies the equation

$$y^{\cdot\cdot} + 2\beta y^{\cdot} + \nu^2 y = \zeta(t); \quad \beta \ll \nu, \quad \nu \approx \Omega \tag{2.11}$$

where $\zeta(t)$ is a stationary centered wide-band random process with spectral density $\Phi_{\zeta\zeta}(\omega)$. Setting

$$y = y_c \cos\nu t + y_s \sin\nu t, \quad y^{\cdot} = \nu(-y_c \sin\nu t + y_s \cos\nu t)$$

and applying the theorem of /7/, we obtain a system of four stochastic Ito's equations in the slow variables $x_{c,s}(t)$, $y_{c,s}(t)$ whose first moments are zero (at least in the case of zero initial conditions). For the second order moments $K_{xcxc} = \langle x_c(t)\, x_c(t) \rangle$, $K_{xcys} = \langle x_c(t)\, y_s(t) \rangle, \ldots$ we have a system of ten equations

$$\begin{aligned}
K^{\cdot}_{xcxc} &= -(2\alpha + q) K_{xcxc} - 2\Delta K_{xcxs} + {}^1/_8\Omega^2 (2D_0 + D_2) K_{xsxs} - \nu^{-1} K_{xcys} \\
K^{\cdot}_{xsxs} &= -(2\alpha + q) K_{xsxs} + 2\Delta K_{xcxs} + \nu^{-1} K_{xsyc} \\
K^{\cdot}_{xcxs} &= -[2\alpha + {}^1/_4\Omega^2 (2D_0 - D_2)] K_{xcxs} + (\Delta - {}^1/_8\Omega^2 E)(K_{xcxs} - K_{xsxs}) + {}^1/_2\nu^{-1}(K_{xcyc} - K_{xsys}) \\
K^{\cdot}_{ycyc} &= -2\beta K_{ycyc} + D_\zeta/\nu^2, \quad K^{\cdot}_{ysys} = -2\beta K_{ysys} + D_\zeta/\nu^2, \quad K_{ycys} = -2\beta K_{ycys} \\
K^{\cdot}_{xcyc} &= -(\alpha + \beta + q) K_{xcyc} - r K_{xsyc} - {}^1/_2\nu^{-1} K_{ycys} \\
K^{\cdot}_{xsys} &= -(\alpha + \beta + q) K_{xsys} + r K_{xcys} + {}^1/_2\nu^{-1} K_{ycys} \\
K^{\cdot}_{xcys} &= -(\alpha + \beta + q) K_{xcys} - r K_{xsys} - {}^1/_2\nu^{-1} K_{ysys} \\
K^{\cdot}_{xsyc} &= -(\alpha + \beta + q) K_{xsyc} + r K_{xcyc} + {}^1/_2\nu^{-1} K_{ycyc}; \\
D_\zeta &= 2\pi\Phi_{\zeta\zeta}(\Omega), \quad q = {}^1/_8\Omega^2 (D_0 - D_2), \quad r = \Delta - {}^1/_{16}\Omega^2 E
\end{aligned} \tag{2.12}$$

A stationary solution of closed form can be obtained for the system of Eqs.(2.12) for a constant $\nu$, from which we obtain

$$\begin{aligned}
\langle A^2 \rangle &= \langle x_c^2 \rangle + \langle x_s^2 \rangle = (D_\zeta / 2\beta\nu^4)(\alpha + \beta + q) \\
&\quad [(\alpha + \beta + q)^2 + r^2]^{-1} \quad (2\alpha - {}^1/_2\Omega^2 D_2)^{-1}
\end{aligned} \tag{2.13}$$

Having determined the quantity $\langle A^2 \rangle / \langle A^2 \rangle_0$ on the basis of (2.13), where $\langle A^2 \rangle_0$ is the value of $\langle A^2 \rangle$ when $\xi(t) \equiv 0$, enables us to draw certain conclusions about the feasibility of reducing vibrations induced by a narrow-band external excitation by random fluctuations of natural frequency. Qualitatively these conclusions are similar to those obtained above for the case of external periodic excitation. However we have in the considered here case the additional effect of the external excitation spectrum width. Assuming in (2.13) the quantities $D_2$, $D_0$, $E$ to be smalls of higher order than $\alpha$, we find that when $\Delta = 0$, the inequality $\langle A^2 \rangle < \langle A^2 \rangle_0$ is satisifed when

$$\Phi_{\xi\xi}(0) > 3\Phi_{\xi\xi}(2\Omega)(1 + {}^2/_3\, \beta/\alpha)$$

The term with $\beta/\alpha$ shows that as the width of external excitation spectrum increases, the effectiveness of the considered method of damping vibrations, as expected, decreases.

Equations (2.8), (2.9) or (2.12) or moments can be used for the numerical solution of the nonstationary problem of passing through resonance in a system with random variation of natural frequency, on condition that the external excitation frequency $\nu$ is a slowly varying function of time.

3. Consider the following system with two degrees of freedom:

$$\begin{aligned}
x_1^{\cdot\cdot} + 2\alpha_1 x_1^{\cdot} + \Omega_1^2 x_1 &= \lambda_{12} x_2 \xi(t) + \zeta_1(t) \\
x_2^{\cdot\cdot} + 2\alpha_2 x_2^{\cdot} + \Omega_2^2 x_2 &= \lambda_{21} x_1 \xi(t) + \zeta_2(t)
\end{aligned} \tag{3.1}$$

where $\xi(t)$, $\zeta_i(t)$ $(i = 1, 2)$ are stationary wide-band centered random processes. Assuming that in (3.1) $\alpha_i$, $\lambda_{ij}$, $\Phi_{\xi\xi}(\omega)$, $\Phi_{\zeta_i\zeta_i}(\omega)$ $(i, j = 1, 2)$ are small, we introduce the substitution of variables

$$x_i = A_i \cos\theta_i, \quad x_i^{\cdot} = -\Omega_i A_i \sin\theta_i, \qquad \theta_i = \Omega_i t + \varphi_i, \quad V_i = A_i^2$$

and apply the method of averaging. As the result we can obtain the following two stochastic equations in Stratonovich's sense:

$$\begin{aligned} &V_1^{\cdot} = [-2\alpha_1 + {}^1/_2\gamma\,(D_+ - D_-)]\,V_1 + {}^1/_2\gamma_1^2(D_+ + D_-)V_2 + \\ &\quad 2D_1 + \gamma_1\,[(2D_+V_1V_2)^{1/2}\eta_+(t) - (2D_-V_1V_2)^{1/2}\eta_-(t)] + \\ &\quad (8D_1V_1)^{1/2}\eta_1(t), \quad V_2^{\cdot} = [-2\alpha_2 + {}^1/_2\gamma\,(D_+ - D_-)]V_2 + \\ &\quad {}^1/_2\gamma_2^2\,(D_+ + D_-)V_1 + 2D_2 + \gamma_2\,[(2D_+V_1V_2)^{1/2}\eta_+(t) + \\ &\quad (2D_-V_1V_2)^{1/2}\eta_-(t)] + (8D_2V_2)^{1/2}\eta_2(t), \quad \gamma_i = \lambda_{ij}/\Omega_i \\ &\gamma = \gamma_1\gamma_2, \quad D_i = (2\pi/\Omega_i^2)\,\Phi_{\zeta\zeta i}(\Omega_i), \quad D_\pm = 2\pi\Phi_{\xi\xi}(\Omega_\pm) \\ &\Omega_\pm = \Omega_2 \pm \Omega_1 \end{aligned} \tag{3.2}$$

where $\eta_\pm(t), \eta_{1,2}(t)$ denotes independent random processes of the white noise type of unit intensity. Passing from (3.2) to respective Ito's equations and applying the operation of mathematic expectation determination with respect to mean squares of amplitude $m_i = \langle V_i \rangle$ which with $\alpha_i, \lambda_{ij}, \Omega_i$ constant has the stationary solution

$$\begin{aligned} &m_i = 2Q^{-1}\,\{D_i\,[2\alpha_j - \gamma\,(D_+ - D_-)] + D_j\gamma_i\,(D_+ + D_-)\} \\ &Q = 2\alpha_1\alpha_2 - \gamma\,(\alpha_1 + \alpha_2)(D_+ - D_-) - 2\gamma^2 D_+D_-, \quad i = 1,2;\; j \neq i \end{aligned} \tag{3.3}$$

This solution has a meaning when the inequality $Q > 0$ which represents the stability condition for system (3.1) in the mean square, is satisfied. Particular variants of that condition that obtain for $D_- = 0, \gamma > 0$ and $D_+ = 0, \gamma < 0$ were derived in /9/ in relation to summary and difference stochastic resonance combinations. Another particular variant of condition $Q > 0$ obtains when $D_+ = D_-$ (mixed summary-difference stochastic combination resonance), coincides with the exact condition of stability of system (3.1) in the mean quadratic, obtained in /10/ by a direct comparison with the equations of moments for system (3.1) with white noise $\xi(t)$ (when $\zeta_i(t) \equiv 0$). Formula (3.3) shows that when $\alpha_1 \neq \alpha_2$, then for $\gamma > 0$ $(\gamma < 0)$ the presence in the spectrum of process $\xi(t)$ of a nonzero components with frequency $\Omega_-(\Omega_+)$ has a stabilizing effect on the excitation of summary (difference) combination resonance. When $\alpha_1 = \alpha_2$, the condition of stability is determined only by one of the quantities $D_+$ or $D_-$ (respectively, for $\gamma > 0$ or $\gamma < 0$) independently of the value of the other of these quantities.

Note that when processes $\xi(t), \zeta_i(t)$ are assumed to be independent white noise the inequality in (3.1) can be used for obtaining a system of 10 equations in second order moments $\langle x_i x_j \rangle$, $\langle x_i x_j^{\cdot} \rangle$, $\langle x_i^{\cdot} x_j^{\cdot} \rangle$, $i, j = 1, 2$ (the corresponding homogeneous system of equations was derived in /10/). A stationary solution of explicit form can be obtained for that system, and, as expected, the equalities $\langle x_i^2 \rangle = {}^1/_2 m_i$ $(i = 1, 2)$, where $m_i$ are determined by formulas (3.3) with $D_+ = D_-$, are valid.

Having composed Ito's equations with respect to processes $u_{ij} = V_iV_j$ using (3.2) and applying the operation of mathematical expectation determination, we obtain for second order equations $K_{ij} = \langle V_iV_j \rangle$ $(i, j = 1, 2)$ of processes $V_i(t)$ the system of three equations

$$\begin{aligned} &K_{11}^{\cdot} = [-4\alpha_1 + 2\gamma\,(D_+ - D_-)]\,K_{11} + 4\gamma_1^2\,(D_+ + D_-)\,K_{12} + 16D_1m_1 \\ &K_{12}^{\cdot} = \gamma_2^2\,(D_+ + D_-)\,K_{11} + [-2\,(\alpha_1 + \alpha_2) + 4\gamma\,(D_+ - D_-)]\,K_{12} + \\ &\quad \gamma_1^2\,(D_+ + D_-)\,K_{22} + 4\,(D_1m_2 + D_2m_1) \\ &K_{22}^{\cdot} = 4\gamma_2^2\,(D_+ + D_-)\,K_{12} + [-4\alpha_2 + 2\gamma\,(D_+ - D_-)]\,K_{22} + 16D_2m_2 \end{aligned} \tag{3.4}$$

The system of Eqs. (3.4) was integrated on a computer for the case of $D_- = 0$ and several combinations of parameters. The results revealed the following regular relationship: in a steady state the normalized correlation coefficient

$$\rho_{12} = \frac{K_{12} - m_1m_2}{(K_{11} - m_1^2)^{1/2}\,(K_{22} - m_2^2)^{1/2}}$$

is a monotonically increasing function of parameter $D_+$, and $\rho_{12} = 0$ when $D_+ = 0$ and $\rho_{12} \to 1$ with the determinant of system (3.4) approaching zero, i.e. with the approach to the stability boundary of system (3.1) with respect to fourth order moments. This proves the theoretical feasibility of identifying the stochastic combination resonance by analyzing the experimentally obtained linear combination $x(t)$ of processes $x_i(t)$. Processes $V_i$ are separated from the recorded realization of $x(t)$ using a pair of band-pass filters with central frequencies $\Omega_1$ and $\Omega_2$. A nonzero $\rho_{12}$ indicates the presence in the system of a combination resonance, and the quantity $1 - \rho_{12}$ may be considered as the stability margin of the system with respect to fourth order moments.

We note in conclusion that the described here method of moments can be readily extended to systems in which the arbitrary parametric perturbations are correlated with external random excitations. In that case the vector equation (1.3) in first order moments contains an additional term independent of $x_i$, containing coefficients of reciprocal intensities of processes of the white noise type $\xi(t)$, $\zeta(t)$.

The author thanks V.B. Kolmanovskii for discussing this paper.

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Translated by J.J.D.